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A hidden-variables versus quantum mechanics experiment

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Abstract. In the light of the Bell–Kochen–Specker theorem, we examine a feasible experimental test in order to discriminate between non-contextual hidden-variables and some probabilistic predictions of quantum mechanics.

1. Introduction

The longstanding question of whether hidden-variable (HV) ‘completions’ of quantum mechanics (QM) are feasible has generated several ‘impossibility’ proofs. One of these, the Bell–Kochen–Specker (BKS) theorem [1, 2], states that any non-contextual HV theory satisfying a few reasonable assumptions is internally inconsistent. As it stands, the BKS theorem is a mathematical statement which does not require any real experiment in order to be proved or refuted. The aim of this paper is to examine the possibility of translating BKS-type arguments into real experiments to test the predictions of QM against some predictions of HV theories satisfying the same assumptions as those considered in the BKS theorem.

The paper is structured as follows. In section 2, we specify the premises which the BKS theorem shows are inconsistent. In section 3, we present a theorem which states that for a single spin-1 particle, any HV theory satisfying BKS premises makes definite predictions on ‘HV values’ of some physical observables. In section 4, we show that these predictions are inconsistent with the corresponding probabilistic predictions of QM (with a stronger discrepancy than in a similar argument considered by Clifton [3]), and we discuss how this can be translated into an experimental test in order to discriminate between QM and any HV theory satisfying BKS premises.

2. Premises in the BKS theorem

Consider an HV theory in which any individual physical system has a set of (hidden) variables the values of which (together with those of the measuring apparatus) determine (in some unspecified way) the outcome of any experiment on that individual system. Suppose these ‘HV values’ satisfy the following premises:

(a) *Non-contextuality*: The HV value of a physical observable A in an individual physical system does not depend on which other observables (compatible with A) have simultaneously defined values in the said individual system.

(b) *Counterfactual definiteness*: An individual physical system can have simultaneously precise HV values for two non-compatible observables, B and β , although these cannot be jointly measured (in the sense of making a joint preparation, or ideal measurement of the first kind [4]).

(c) *Constraints on HV values suggested by QM:* (c1) If the measure of a physical observable A in an ensemble of similarly prepared systems gives results only on a set of discrete real values (which, according to QM, belong to the spectrum of the associated Hermitian operator \hat{A}), the HV value of such an observable, which we shall denote by $v(A)$, must be one of such values in any individual physical system of the ensemble.

Corollary. If a quantum mechanical state is dispersion-free for an observable A (which, according to QM, means that it is mathematically describable by an eigenstate of \hat{A} corresponding to some eigenvalue a), any of the individual systems in the said state will have an HV value for A equal to the eigenvalue,

$$v(A) = a. \quad (1)$$

(c2) Any set of HV values corresponding to a set of compatible observables (in QM described by commuting Hermitian operators) satisfies the same relations as any possible outcome of measuring those observables according to QM. Namely: let $\{A, B, C, \dots\}$ be a set of pairwise compatible observables, and let us assume that the Hermitian operators which represent them satisfy a certain functional identity

$$f(\hat{A}, \hat{B}, \hat{C}, \dots) = 0. \quad (2)$$

The outcome of a joint measurement of this set of observables on an individual system will be (according to QM) a set of eigenvalues of the corresponding operators $\{a, b, c, \dots\}$, satisfying the relation

$$f(a, b, c, \dots) = 0. \quad (3)$$

Assumption (c2) states that any set of HV values for these observables must verify the same relation

$$f(v(A), v(B), v(C), \dots) = 0. \quad (4)$$

Premises (c1), (c2), although reasonable, are far from being inevitable, even for HV theories with predictions that are fully compatible with those of QM. For example, in the Dewdney *et al* [5, 6] discussion of spin- $\frac{1}{2}$ particles, according to Bohm's theory [7], the three spin components of an individual particle have simultaneously well-defined continuous (hidden) values; it is in the interaction with a measuring device, as a Stern-Gerlach apparatus, for instance, that the discretization of the measured spin component appears (in such a way that the existence of ensembles that are dispersion-free for more than one spin component is forbidden). The point is that, in order to obtain an 'impossibility' proof like BKS's, one must accept some premises which not all HV theories satisfy; Bohm's HV theory is compatible with orthodox QM but does not satisfy either assumptions (a) or (c) [6] and therefore is not affected by BKS's theorem.

3. A BKS-like theorem for a spin-1 particle

The BKS theorem states that the premises considered in section 2 are inconsistent. The usual way of demonstrating this inconsistency is by explicitly constructing a set of observables in such a way that any assignation of HV values according to such premises becomes

impossible. This is what we will call a ‘non-probabilistic’ state-independent proof of the BKS theorem. Examples of this kind are the original proof by Kochen and Specker [2] (recently discussed and simplified in [8]) and those obtained from impossibility proofs of local HV theories [9–13] which also admit a reading as BKS proofs [14, 15]. However, some authors, invoking parts of some of the ‘non-probabilistic’ proofs, have constructed ‘probabilistic’ arguments which point out discrepancies between definite predictions of HV theories and probabilistic predictions of QM. This possibility was first suggested by Stairs [16] (considering an eight-direction lemma in Kochen–Specker’s original proof [2]), and has been more explicitly developed by Clifton [3] (who used a 13-direction argument in Bell’s proof [1] as well as Kochen–Specker’s same eight-direction lemma). In the following paragraphs we present a different construction with 14 directions and stronger discrepancies.

Consider a single spin-1 particle, and let S_a be the spin component in a direction a ; according to (c1), $v(S_n^2) \in \{0, \hbar^2\}$. Let us suppose that b and c are two mutually orthogonal directions both orthogonal to a , and let b' and c' be another two. According to (a), (b) and (c2),

$$\begin{aligned} v(S_a^2) &= 2\hbar^2 - v(S_b^2) - v(S_c^2) \\ &= 2\hbar^2 - v(S_{b'}^2) - v(S_{c'}^2) \end{aligned} \tag{5}$$

since, according to (c2), HV values $v(S_n^2)$ along three orthogonal directions satisfy the same relation as the square of spin components, $\hat{S}_a^2 + \hat{S}_b^2 + \hat{S}_c^2 = 2\hbar^2$; according to (b), different pairs of orthogonal directions b, c can have simultaneously well-defined values (although the observables S_b^2 and S_c^2 cannot be measured jointly with $S_{b'}^2$ and $S_{c'}^2$); and according to (a), the HV value in the direction orthogonal to both, $v(S_a^2)$, is the same whatever the election of b, c .

Theorem. If $v(S_z^2) = 0$, and

$$\gamma \leq \theta \leq \pi - \gamma \quad \gamma := \arctan[4(3^{-3/4})] \approx 60^\circ 19' 20'' \tag{6}$$

any HV theory satisfying conditions (a), (b), (c1) and (c2) predicts with certainty that $v(S_\varphi^2) = \hbar^2$, where S_φ stands for a spin component in any direction that forms an angle θ with the z axis.

For the sake of simplicity, from now on, $v(n)$ will represent the value $v(S_n^2)$, and ‘ \perp ’ the constraint (5), which establishes that one of the HV values for three mutually orthogonal directions must be zero and the other two values must be \hbar^2 .

Proof. By *reductio ad absurdum*. Let $n = (0, 0, 1)$, $n' = (\sin\theta, 0, \cos\theta)$, $\theta \in (0, \pi)$ and let us assume that $v(n)$ and $v(n')$ are both zero. If $v(0, 0, 1) = 0$, then (5) implies that the value in any direction orthogonal to n must be \hbar^2 †; specifically,

$$v(0, 0, 1) = 0 \stackrel{\perp}{\implies} \begin{cases} v(1, 0, 0) = \hbar^2 \\ v(\cos\varphi_k, \sin\varphi_k, 0) = \hbar^2 \end{cases} \quad k = 1, 2, 3 \tag{7}$$

† These definite HV values for an infinite set of non-compatible observables do not contradict QM predictions: According to QM, in the eigenstate of S_z with zero eigenvalue, any S_b^2 with b in the $z = 0$ plane, if measured, will give a result \hbar^2 with certainty, even if two such observables S_b^2, S_c^2 are not jointly measurable except when $b \perp c$. This is a kind of QM counterfactual definiteness (another more familiar example is the singlet state of two spin- $\frac{1}{2}$ particles: the observable $S_n^{(1)} + S_n^{(2)}$, if measured, will give a zero result $\forall n$).

where we shall choose $\varphi_1 \neq \varphi_2 \neq \varphi_3 \neq n\pi/2$ (with n integer).

Also from (5)

$$v(\sin \theta, 0, \cos \theta) = 0 \stackrel{\perp}{\Rightarrow} v(-1, \cot \varphi_k, \tan \theta) = \hbar^2 \quad k = 1, 2. \quad (8)$$

(The directions on the right-hand side of (8), as well as in subsequent equations, have not been normalized, as this is irrelevant to the argument and would only complicate the expression.) The directions that appear on the lower right-hand side in (7) and on the right-hand side in (8) are mutually orthogonal; therefore the value in a direction orthogonal to both must be zero,

$$\left. \begin{aligned} v(\cos \varphi_k, \sin \varphi_k, 0) &= \hbar^2 \\ v(-1, \cot \varphi_k, \tan \theta) &= \hbar^2 \end{aligned} \right\} \stackrel{\perp}{\Rightarrow} v(\sin \varphi_k, -\cos \varphi_k, \cot \theta \operatorname{cosec} \varphi_k) = 0 \quad k = 1, 2. \quad (9)$$

According to (5), the value in any direction orthogonal to any of the two on the right-hand side in (9) must be \hbar^2 ; in particular,

$$v(\sin \varphi_1, -\cos \varphi_1, \cot \theta \operatorname{cosec} \varphi_1) = 0 \stackrel{\perp}{\Rightarrow} v(0, \cot \theta, \sin \varphi_1 \cos \varphi_1) = \hbar^2 \quad (10)$$

and

$$v(\sin \varphi_2, -\cos \varphi_2, \cot \theta \operatorname{cosec} \varphi_2) = 0 \stackrel{\perp}{\Rightarrow} v(-\sin \varphi_3, \cos \varphi_3, \tan \theta \sin \varphi_2 \cos(\varphi_2 - \varphi_3)) = \hbar^2. \quad (11)$$

The direction that appears on the right-hand side in (10) is orthogonal to $(1, 0, 0)$ which has value \hbar^2 in (7), so

$$\left. \begin{aligned} v(1, 0, 0) &= \hbar^2 \\ v(0, \cot \theta, \sin \varphi_1 \cos \varphi_1) &= \hbar^2 \end{aligned} \right\} \stackrel{\perp}{\Rightarrow} v(0, -\sin \varphi_1 \cos \varphi_1, \cot \theta) = 0. \quad (12)$$

Analogously, as the direction that appears on the right-hand side in (11) is orthogonal to $(\cos \varphi_3, \sin \varphi_3, 0)$ which have value \hbar^2 in (7),

$$\left. \begin{aligned} v(\cos \varphi_3, \sin \varphi_3, 0) &= \hbar^2 \\ v(-\sin \varphi_3, \cos \varphi_3, \tan \theta \sin \varphi_2 \cos(\varphi_2 - \varphi_3)) &= \hbar^2 \end{aligned} \right\} \stackrel{\perp}{\Rightarrow} v(\sin \varphi_3, -\cos \varphi_3, \cot \theta \operatorname{cosec} \varphi_2 \sec(\varphi_2 - \varphi_3)) = 0. \quad (13)$$

The directions on the right-hand side in (12) and (13) are mutually orthogonal if

$$-\sin(2\varphi_1) \sin \varphi_2 \cos \varphi_3 \cos(\varphi_2 - \varphi_3) = 2 \cot^2 \theta. \quad (14)$$

The left-hand side of (14) is bounded between $-3\sqrt{3}/8$ ($\varphi_1 = \pi/4, \varphi_2 = \pi/3, \varphi_3 = \pi/6$) and $3\sqrt{3}/8$, so that in order for directions at the right in (12) and (13) to be orthogonal, the following requirement must be fulfilled:

$$\arctan[4(3^{-3/4})] \leq \theta \leq \arctan[-4(3^{-3/4})]. \quad (15)$$

In short: if we choose $\theta, \varphi_1, \varphi_2$ and φ_3 satisfying (14), the two directions on the right-hand side in (12) and (13) are orthogonal and the corresponding HV values are both zero. But, according to (5), the HV values for two orthogonal directions cannot both be zero. So we have reached a contradiction; the only way of avoiding it is for the initial hypothesis, $v(\mathbf{n}) = v(\mathbf{n}') = 0$, to be false. \square

In this proof, the interval $\gamma \leq \theta \leq \pi - \gamma$ for which $v(S_z^2) = 0 \Rightarrow v(S_\theta^2) = \hbar^2$ has reached $\gamma \approx 60^\circ 19' 20''$ assuming definite HV values $v(\mathbf{n})$ for observables S_n^2 along 14 directions (against $\gamma \approx 70^\circ 31' 44''$ with 8 directions, and $\gamma \approx 63^\circ 26' 6''$ with 13 directions).

4. Experimental test

If what we are looking for is an experimental test which will discriminate between QM and an HV theory built up around premises (a), (b), (c1) and (c2) leading to the non-probabilistic BKS theorem, we need to clarify the relationship between the ‘HV values’ and the outcomes of measurements. Usually the following property is assumed.

(d) *Faithful measurement*: If an individual system has a hidden value $v(A)$ for an observable A , a measurement of A will give the result $v(A)$.

This assumption is not necessary for a proof of the non-probabilistic BKS theorem and is experimentally unverifiable (if there is a dispersion in the results, only the frequencies of predictions and results can be compared). Moreover, it is not satisfied by all HV theories (Bohm’s theory [5–7] is again a counterexample). Fortunately, for an experimental test of our family of HV theories, a weaker assumption will do:

(d’) If an HV theory assigns the same value $v(A)$ to any individual system of an ensemble, a measurement of A in any of the systems of the ensemble will give the result $v(A)$.

Unlike (d), assumption (d’) is experimentally verifiable. Note that (d’) and the corollary of (c1) assume the equality of HV values and QM results only in cases when both are dispersion-free in some ensemble of systems; this is true even in Bohm’s theory.

Let us now analyse the different results predicted by HV theories and QM in the case considered in section 3. If we prepare an ensemble of spin-1 particles in a quantum dispersion-free state with zero value for S_z , from premise (c1) all individual systems of the ensemble verify

$$v(S_z^2) = s_z^2 = 0. \tag{16}$$

For any individual physical system with this HV value, as we have shown, any HV theory satisfying (a), (b), (c) predicts *with certainty*

$$v(S_\theta^2) = \hbar^2 \tag{17}$$

if the angle θ verifies (6). Therefore, according to (d’), the HV prediction is that the result of a measurement of S_θ^2 in any member of the ensemble will be \hbar^2 . On the other hand, QM predicts that, if we measure the observable S_θ on the ensemble of individual systems in the $s_z = 0$ state, the probability of getting the result $s_\theta = 0$ ($\Rightarrow s_\theta^2 = 0$) is

$$P_{|s_z=0\rangle}(s_\theta = 0) = |\langle s_\theta = 0 | s_z = 0 \rangle|^2 = \cos^2 \theta. \tag{18}$$

Therefore both theories predict conflicting results for the same experiment. The largest discrepancy with the HV prediction is obtained when

$$\theta = \gamma := \arctan[4(3^{-3/4})] \approx 60^\circ 19' 20'' \Rightarrow P_{|s_z=0\rangle}(s_\gamma = 0) = \left(1 + \frac{16}{\sqrt{27}}\right)^{-1} \approx \frac{1}{4}. \tag{19}$$

That is to say: QM predicts that almost one out of four times the result will be $s_\gamma = 0$, an outcome which is incompatible with the HV value $v(S_\gamma^2) = \hbar^2$ plus assumption (d’). (The probability for a contradiction between QM and HV predictions reaches 1/5 with 13 directions, and 1/9 with 8 directions, as [17] and [18] correctly remark.)

The preparations and measures involved in our discussion can be made by means of traditional Stern–Gerlach devices, instead of with more complicated measures of quadrupolar moments, as in [19]. The explicit angular dependence of the QM prediction, (18), should facilitate the discrimination between signal and noise in a real experiment.

The simple experiment described above would (presumably!) allow us to confirm the quantum prediction, and therefore to *experimentally* exclude the existence of any HV theory satisfying conditions (a), (b), (c1), (c2) and (d'). In the search for HV alternatives to QM, at least one of these premises should be abandoned.

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